

MODELIZATION AND EXTRAPOLATION OF TIME DEVIATION OF USO AND ATOMIC CLOCKS IN GNSS-2 CONTEXT

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Abstract

In Global Navigation Satellite Systems (GNSS), the on-board time has to be modeled and predicted in order to broadcast the time parameters to final users. As a consequence, the time prediction performance of the on-board clocks has to be characterized.

In order to estimate the time uncertainty of the on-board oscillator, a linear or parabolic fit is performed over the sequence of observed time difference and extrapolated during the prediction period. In 1998 the Centre National d'Etudes Spatiales (CNES) proposed specifications of orbit determination and time synchronization for GNSS-2. The needs of synchronization were specified as the maximum error of the time difference prediction from the extrapolated fit.

Using our work about the estimation of uncertainties in time error extrapolation, we have translated these time domain specifications into a noise level limit or an Allan deviation limit. Of course, these limits depend on the main type of noise for integration time of about 1 day and on the type of adjustment which is performed (linear for cesium clocks and quadratic for other oscillators).

A table summarizing these limits is presented. These values are compared to experimental results obtained with different types of oscillators (quartz, rubidium, and cesium).

INTRODUCTION

In Global Navigation Satellite Systems, the on-board time has to be modeled and predicted in order to broadcast the time parameters to final users. As a consequence,

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the time prediction performance of the on-board clocks has to be characterized.

In order to estimate the time uncertainty of the on-board oscillator, a linear or parabolic fit is performed over the sequence of observed time difference and extrapolated during the prediction period, i. e. when the satellite is out of visibility. In 1998, CNES proposed specifications of orbit determination and time synchronization for GNSS-2. The needs of synchronization were specified as the maximum error of the time difference prediction from the extrapolated fit.

The question is then: how is this maximum error related to the noise levels of the clock? Despite several papers deal with this question [1, 2, 3, 4, 5], a new approach was chosen here because we are not only interested in the asymptotic trend of this maximum deviation, but also in its evolution close to the interpolated sequence.

In this paper, we will call Time Interval Error (TIE) the differences between the extrapolated parabola and the real time deviation $x(t)$. By definition, the TIE samples are then the residuals to this parabola. However, in the following, we will limit the use of the word “residuals” to the differences between the interpolated parabola and the real time deviation $x(t)$.

The TIE is due to two effects: the error of determination of the parabolic parameters and the error due the noise of the clock. Obviously, both of these errors may be positive or negative, and the ensemble average of the TIE is equal to zero. Moreover, it can be easily shown that the statistics of the TIE is Gaussian. Consequently, we only have to estimate the variance of the TIE in order to completely define its statistical characteristics.

Moreover, the removal of a quadratic fit from the time deviation sequence cancels out the non-stationarity problem of very low frequency noises [6, 7], and the variance of the TIE (i.e. the “true variance”) converges for all types of noise without considering a hypothetical low cut-off frequency.

In order to determine an estimation of the TIE, we will already redefine the interpolation method. Then, we will compare the equations giving the theoretical estimates of the variance of the TIE to simulations and to real data.

ASSESSMENT PRINCIPLE OF THE INTERPOLATION AND EXTRAPOLATION ERRORS

Interpolating Functions

Rather than performing a classical linear or quadratic fit, we used the Tchebychev polynomials [8, 7]:

$$x(t) = P_0\Phi_0(t) + P_1\Phi_1(t) + P_2\Phi_2(t) + e(t). \quad (1)$$

where $e(t)$ is the noise, i.e. the purely random behavior of $x(t)$, $\{\Phi_0(t), \Phi_1(t), \Phi_2(t)\}$ are the first 3 Tchebychev polynomials and the parameters $\{P_0, P_1, P_2\}$ have the same dimension as $x(t)$, i.e. time.

Estimation of the Residuals

The variance of the residuals σ_e^2 may be estimated by [8, 7]:

$$\sigma_e^2 = \sigma_X^2 - \frac{1}{N} (\sigma_{P_0}^2 + \sigma_{P_1}^2 + \sigma_{P_2}^2). \quad (2)$$

Correlation of the Samples

Obviously, the long-term behavior of the TIE depends greatly on the type of noise. The autocorrelation function $R_x(t)$ of the $x(t)$ data contains the information about this type of noise. The power spectral density (PSD) $S_x(f)$ is the Fourier Transform of the autocorrelation function $R_x(t)$.

Taking into account f_l , the low cut-off frequency and f_h , the high cut-off frequency, the autocorrelation function may be written [7]:

$$R_x(t_j - t_i) = \int_{f_l}^{f_h} S_x(f) \cos [2\pi f(t_j - t_i)] df. \quad (3)$$

We will use the power law model of $S_x(f)$:

$$S_x(f) = \sum_{\alpha=-4}^0 k_\alpha f^\alpha. \quad (4)$$

Calculation Method of the TIE

By hypothesis, we consider that the TIE is the difference between the true time deviation $x(t)$ at time t , and the extrapolation of the parabola (previously estimated from t_0 to t_{N-1}) up to this time $t > t_{N-1}$:

$$\text{TIE}(t) = x(t) - \hat{P}_0\Phi_0(t) - \hat{P}_1\Phi_1(t) - \hat{P}_2\Phi_2(t) \quad \text{and} \quad t > t_{N-1}. \quad (5)$$

Thus, the quadratic ensemble average of TIE may be estimated by:

$$\begin{aligned} \langle \text{TIE}^2(t) \rangle &= \langle x^2(t) \rangle + \langle \hat{P}_0^2 \rangle \Phi_0^2(t) + \langle \hat{P}_1^2 \rangle \Phi_1^2(t) + \langle \hat{P}_2^2 \rangle \Phi_2^2(t) \\ &\quad - 2 \left[\langle x(t) \hat{P}_0 \rangle \Phi_0(t) + \langle x(t) \hat{P}_1 \rangle \Phi_1(t) + \langle x(t) \hat{P}_2 \rangle \Phi_2(t) \right] \\ &\quad + 2 \left[\langle \hat{P}_0 \hat{P}_1 \rangle \Phi_0(t) \Phi_1(t) + \langle \hat{P}_0 \hat{P}_2 \rangle \Phi_0(t) \Phi_2(t) + \langle \hat{P}_1 \hat{P}_2 \rangle \Phi_1(t) \Phi_2(t) \right]. \end{aligned} \quad (6)$$

Consequently, for each type of noise, we have to know:

- $\langle x^2(t) \rangle = R_x(t)$, the autocorrelation function of $x(t)$;
- the 3 variances $\langle \hat{P}_i^2 \rangle = \sigma_{P_i}^2$;
- the 3 covariances $\langle \hat{P}_i \hat{P}_j \rangle = \text{Cov}(P_i, P_j)$: actually, only $\langle \hat{P}_0 \hat{P}_2 \rangle \neq 0$;
- the 3 covariances $\langle x(t) \hat{P}_i \rangle = \text{Cov}(x(t), P_i)$.

THEORETICAL RESULTS

Only the final results are given here, but the calculation details are available upon request to the authors.

In the following, T_m designates the measurement time, i. e. the duration of the interpolated sequence:

$$T_m = N\tau_0. \quad (7)$$

and T_p is the prediction time whose origin is the beginning of the extrapolated sequence (and the end of the interpolated sequence).

Case of a Quadratic Interpolation

Variance of the residuals

From (2) we got the following results versus $T_m = N\tau_0$:

- White FM :

$$\sigma_e^2(T_m) \approx \frac{3\pi^2 k_{-2} T_m}{35}. \quad (8)$$

- Flicker FM:

$$\sigma_e^2(T_m) \approx \frac{\pi^2 k_{-3} T_m^2}{24}. \quad (9)$$

- Random walk FM:

$$\sigma_e^2(T_m) \approx \frac{\pi^4 k_{-4} T_m^3}{315}. \quad (10)$$

All the above equations were obtained under the assumption $N \gg 1$, i. e. $T_m \gg \tau_0$.

Estimation of the TIE using the noise levels

The theoretical calculation of (6) yields the following variances versus the measurement time T_m and the prediction time T_p :

- White FM:

$$\langle \text{TIE}^2(T_m, T_p) \rangle \approx \frac{6\pi^2 k_{-2} T_m}{35} \left(50 \frac{T_p^4}{T_m^4} + 100 \frac{T_p^3}{T_m^3} + 69 \frac{T_p^2}{T_m^2} + 19 \frac{T_p}{T_m} + 1 \right). \quad (11)$$

- Flicker FM:

$$\begin{aligned} \langle \text{TIE}^2(T_m, T_p) \rangle \approx & \frac{\pi^2 k_{-3} T_m^2}{8} \left[192 \frac{T_p^6}{T_m^6} + 576 \frac{T_p^5}{T_m^5} + 692 \frac{T_p^4}{T_m^4} + 424 \frac{T_p^3}{T_m^3} + 136 \frac{T_p^2}{T_m^2} + 20 \frac{T_p}{T_m} + 1 \right. \\ & \left. + 96 \frac{T_p^3}{T_m^3} \ln \left(\frac{T_p}{T_m + T_p} \right) \left(2 \frac{T_p^4}{T_m^4} + 7 \frac{T_p^3}{T_m^3} + 9 \frac{T_p^2}{T_m^2} + 5 \frac{T_p}{T_m} + 1 \right) \right]. \quad (12) \end{aligned}$$

- Random walk FM:

$$\langle \text{TIE}^2(T_m, T_p) \rangle \approx \frac{2\pi^4 k_{-4} T_m^3}{315} \left(450 \frac{T_p^4}{T_m^4} + 690 \frac{T_p^3}{T_m^3} + 303 \frac{T_p^2}{T_m^2} + 42 \frac{T_p}{T_m} + 2 \right). \quad (13)$$

All the above equations were obtained under the assumption $N \gg 1$. Thus, under this assumption, the dependence versus N cancels out: the variance of the residuals only depends on the length of the interpolated sequence T_m , and the variance of the TIE only depends on the ratio $\frac{T_p}{T_m}$, whatever the number of samples N is.

Case of a Linear Interpolation

Cesium clocks are not affected by a quadratic drift. If such clocks are used, we may limit the fit to a linear interpolation. In this case, for long term, the main contribution to the TIE will be due to the error on the parameter P_1 and then will be lower than for a quadratic interpolation. It is then strongly recommended to extrapolate the time deviation of a cesium clock from a linear fit. On the other hand, the variance of the residuals will be higher because a quadratic adjustment remains closer to the time data than a linear one.

Variance of the residuals

- White FM:

$$\sigma_e^2(T_m) \approx \frac{2\pi^2 k_{-2} T_m}{15} \quad (14)$$

- Flicker FM:

$$\sigma_e^2(T_m) \approx \frac{\pi^2 k_{-3} T_m^2}{9} \quad (15)$$

- Random walk FM¹:

$$\sigma_e^2(T_m) \approx \frac{2\pi^4 k_{-4} T_m^3}{105} \quad (16)$$

All the above equations were obtained under the assumption $N \gg 1$, i. e. $T_m \gg \tau_0$.

Estimation of the TIE using the noise levels

- White FM:

$$\langle \text{TIE}^2(T_m, T_p) \rangle \approx \frac{4\pi^2 k_{-2} T_m}{15} \left(9 \frac{T_p^2}{T_m^2} + 9 \frac{T_p}{T_m} + 1 \right). \quad (17)$$

- Flicker FM:

$$\langle \text{TIE}^2(T_m, T_p) \rangle \approx \frac{\pi^2 k_{-3} T_m^2}{3} \left[12 \frac{T_p^4}{T_m^4} + 24 \frac{T_p^3}{T_m^3} + 20 \frac{T_p^2}{T_m^2} + 8 \frac{T_p}{T_m} + 1 \right]$$

¹The random walk FM is treated here for homogeneity, but a cesium clock is never affected by this type of noise.

$$\begin{aligned}
& +2 \ln \left(1 + \frac{T_p}{T_m} \right) \left(6 \frac{T_p^2}{T_m^2} + 6 \frac{T_p}{T_m} + 1 \right) \\
& + 2 \frac{T_p^3}{T_m^3} \ln \left(\frac{T_p}{T_m + T_p} \right) \left(6 \frac{T_p^2}{T_m^2} + 15 \frac{T_p}{T_m} + 8 \right) \Big]. \quad (18)
\end{aligned}$$

- Random walk FM¹:

$$\langle \text{TIE}^2(T_m, T_p) \rangle \approx \frac{8\pi^4 k_{-4} T_m^3}{105} \left(35 \frac{T_p^3}{T_m^3} + 39 \frac{T_p^2}{T_m^2} + 11 \frac{T_p}{T_m} + 1 \right). \quad (19)$$

All the above equations were obtained under the assumption $N \gg 1$, i.e. $T_m \gg \tau_0$.

Estimation of the TIE Using the Variance of the Residuals

The relationships (11) to (13) and (17) to (19) needs an explicit knowledge of the noise levels k_α . However, for very-long-term interpolation (several days), we may be sure of the dominant type of noise: the flicker FM for a cesium clock or the random walk FM for a quartz oscillator.

Thus, the variance of the TIE may be directly estimated from the variance of the residuals.

As an example, the variance of the TIE for a random walk FM and a quadratic fit may be rewritten from (13) and (10):

$$\langle \text{TIE}^2(T_m, T_p) \rangle \approx 2\sigma_e^2 \left(450 \frac{T_p^4}{T_m^4} + 690 \frac{T_p^3}{T_m^3} + 303 \frac{T_p^2}{T_m^2} + 42 \frac{T_p}{T_m} + 2 \right). \quad (20)$$

However, such a method is less precise than the use of a correct estimation of the noise levels. This is due to the statistics of the estimate of the variance of the residuals (see section “CONFIDENCE INTERVALS”).

Relationships With the Time Variance

The Time Variance (TVAR), and its square root, the Time Deviation (TDEV), are commonly used for time analysis [9]. Table 1 gives the relationships between the variance of the residuals σ_e^2 and TVAR(τ) for an integration time $\tau = T_m = N\tau_0$. It is interesting to notice that, for a given type of noise, the ratio $\sigma_e^2/\text{TVAR}(\tau)$ is constant.

EXPERIMENTAL VALIDATION

Monte-Carlo Simulations

In order to verify the equations (11) to (13) and (17) to (19), we simulated time deviation sequences of different types of noise (white FM, flicker FM, and random walk FM) and we used quadratic and linear fits. For each type of noise, 10,000 realizations were calculated with

- the same noise level: $k_{-2} = 1.4 \cdot 10^{-4}\text{s}$, $k_{-3} = 3.3 \cdot 10^{-8}$ or $k_{-4} = 5.0 \cdot 10^{-12}\text{s}^{-1}$ for the quadratic fit, $k_{-2} = 3.5 \cdot 10^{-3}\text{s}$, $k_{-3} = 4.8 \cdot 10^{-7}$ or $k_{-4} = 3.3 \cdot 10^{-11}\text{s}^{-1}$ for the linear fit,

| $S_x(f)$ | $k_{-2}f^{-2}$ | $k_{-3}f^{-3}$ | $k_{-4}f^{-4}$ |
|--------------------------------------|----------------------------------|---|-------------------------------------|
| $\text{TVAR}(\tau)$ | $\frac{\pi^2 k_{-2} \tau}{3}$ | $\frac{\pi^2 [27 \ln(3) - 32 \ln(2)] k_{-3} \tau^2}{6}$ | $\frac{11 \pi^4 k_{-4} \tau^3}{15}$ |
| Quadratic interpolation | | | |
| σ_e^2 | $\frac{3 \pi^2 k_{-2} \tau}{35}$ | $\frac{\pi^2 k_{-3} \tau^2}{24}$ | $\frac{\pi^4 k_{-4} \tau^3}{315}$ |
| $\frac{\sigma_e^2}{\sigma_e}$ | $\frac{9}{35}$ | $\frac{1}{4}$ | $\frac{1}{231}$ |
| $\frac{\text{TVAR}(\tau)}{\sigma_e}$ | $\frac{35}{35}$ | $\frac{4 [27 \ln(3) - 32 \ln(2)]}{0.18}$ | $\frac{231}{0.066}$ |
| $\frac{\text{TDEV}(\tau)}{\sigma_e}$ | 0.51 | 0.18 | 0.066 |
| Linear interpolation | | | |
| σ_e^2 | $\frac{2 \pi^2 k_{-2} \tau}{15}$ | $\frac{\pi^2 k_{-3} \tau^2}{9}$ | $\frac{6 \pi^4 k_{-4} \tau^3}{315}$ |
| $\frac{\sigma_e^2}{\sigma_e}$ | $\frac{2}{5}$ | $\frac{2}{3}$ | $\frac{2}{77}$ |
| $\frac{\text{TVAR}(\tau)}{\sigma_e}$ | $\frac{5}{0.63}$ | $\frac{3 [27 \ln(3) - 32 \ln(2)]}{0.30}$ | $\frac{77}{0.16}$ |
| $\frac{\text{TDEV}(\tau)}{\sigma_e}$ | 0.63 | 0.30 | 0.16 |

Table 1: Comparison between the Time Variance $\text{TVAR}(\tau)$ and the variance of the residuals σ_e^2 for different types of noise and for quadratic and linear interpolation.

- the same number of data: 65,536,
- the same number of data taken into account for the fit: $N = 8640$,
- the same time of estimation of the TIE: $\frac{T_m + T_p}{\tau_0}$ equals to $\{8640, 9900, 11350, 13000, 14900, 17000, 19500, 22400, 25700, 29400, 33700, 38600, 44300, 50700, 58100, 65535\}$.

The noise levels were chosen for getting a variance of the residuals equal to 1.

Figure 1 shows the curves corresponding to the square root of equations (11) to (13) and (17) to (19) compared to the standard deviation estimated from the 10,000 simulations. The simulations exhibit a quite good agreement with the theoretical curves.

The asymptotic behaviors are reached at the end of the log-log graphs. The benefits of the linear interpolation are obvious.

Real Clocks (Quartz USO and Atomic Clocks)

Figure 2 compares the long-term behavior of a real ultra-stable quartz oscillator (denoted “quartz 1” in table 2) to the bounds given by the estimated standard deviation of the TIE (the square root of Equations (11) to (13)).

The data from the oscillator are time deviations sampled with a sampling period $\tau_0 = 10$ s, obtained at CNES Toulouse (France) with their own reference clocks (Cs HP 5071A option 001 and H-maser EFOS-16).

Allan variance revealed that 2 types of noise must be taken into account: white FM ($h_0 = 10^{-22}$ s, i.e. $k_{-2} = 2,5 \cdot 10^{-24}$ s) and random walk FM ($h_{-2} = 1,5 \cdot 10^{-31}$ s⁻¹, i.e.

| Clock | $\sigma_y(24h)$ | h_{-2} | h_{-1} | h_0 | σ_e exp. | σ_e theo. | σ_{TIE} exp. | σ_{TIE} theo. (from h_α) | σ_{TIE} theo. (from σ_e) |
|----------|----------------------|----------------------|----------------------|----------------------|--------------------|---------------------|------------------------|---|---|
| | | (s ⁻¹) | | (s) | (ns) | (ns) | (ns) | (ns) | (ns) |
| Quartz 1 | $1.8 \cdot 10^{-13}$ | 0 | $2.2 \cdot 10^{-26}$ | $7.5 \cdot 10^{-23}$ | 1.4 | 1.4 | 7.1 | 6.2 | 6.6 |
| Quartz 2 | $2.9 \cdot 10^{-12}$ | $1.4 \cdot 10^{-29}$ | $1.6 \cdot 10^{-25}$ | 0 | 9.6 | 9.2 | 56.5 | 52.0 | 56.0 |
| Quartz 3 | $3.0 \cdot 10^{-12}$ | $1.4 \cdot 10^{-29}$ | $6.4 \cdot 10^{-25}$ | 0 | 10.2 | 11.0 | 55.7 | 59.0 | 59.5 |
| Rb 1 | $2.7 \cdot 10^{-13}$ | $1.2 \cdot 10^{-31}$ | 0 | $5.3 \cdot 10^{-22}$ | 1.2 | 1.3 | 6.2 | 5.7 | 7.1 |
| Cs 1 | $9.2 \cdot 10^{-14}$ | 0 | 0 | $1.5 \cdot 10^{-21}$ | 1.7 | 1.6 | 5.5 | 5.5 | 5.6 |
| Cs 2 | $3.0 \cdot 10^{-14}$ | 0 | $2.1 \cdot 10^{-28}$ | $1.1 \cdot 10^{-22}$ | 0.6 | 0.5 | 1.9 | 1.6 | 3.0 |

Table 2: Comparison between measured and estimated values of σ_e and σ_{TIE} for several clocks with a quadratic interpolation.

| Clock | $\sigma_y(24h)$ | h_{-1} | h_0 | σ_e exp. | σ_e theo. | σ_{TIE} exp. | σ_{TIE} theo. (from h_α) | σ_{TIE} theo. (from σ_e) |
|-------|----------------------|----------------------|----------------------|--------------------|---------------------|------------------------|---|---|
| | | | (s) | (ns) | (ns) | (ns) | (ns) | (ns) |
| Cs 1 | $9.2 \cdot 10^{-14}$ | 0 | $1.5 \cdot 10^{-21}$ | 2.0 | 2.1 | 4.1 | 4.6 | 4.5 |
| Cs 2 | $3.0 \cdot 10^{-14}$ | $2.1 \cdot 10^{-28}$ | $1.1 \cdot 10^{-22}$ | 0.8 | 0.6 | 1.5 | 1.4 | 1.8 |

Table 3: Comparison between measured and estimated values of σ_e and σ_{TIE} for two cesium clocks with a linear interpolation.

$k_{-4} = 4 \cdot 10^{-33} \text{ s}^{-1}$). Thus, the bounds of Figures 2 were obtained by using the square root of the sum of (11) and (13). This fit was performed over a 24-hour sequence, but the interpolated sequence was shifted along the 90-hour data sequence. At each step ($\tau_0 = 10\text{s}$) of this shift, the fit was extrapolated over 3.5 hours. The TIE measured at this instant ($T_p = 3.5$ hours) was plotted (solid line) and the interpolated sequence was shifted again.

The TIE bounds of the left figure (dashed lines) were estimated from the noise levels as in Figure 2. In the right figure, the TIE bounds were estimated from the variance of the residuals, which was calculated at each step of the shift. In this case, we assumed that the random walk FM was dominant and we used the square root of (20).

The experimental TIE curves remain generally in the theoretical bounds. The few moments where the TIE is outside the bounds is fully compatible with the statistics of TIE (see next section).

Table 2 shows experimental results obtained with several clocks and a quadratic interpolation. The noise levels were estimated from Allan variance measurements. This table compares “ σ_e exp.”, the standard deviation of the residuals, with “ σ_e theo.”, the estimate of σ_e obtained from the square root of (8), (9), (10) and the noise level coefficients.

This table compares also “ σ_{TIE} exp.”, the measured error between the parabola extrapolated over 3.5 hours and the real-time deviation of the clock at this instant, with

“ σ_{TIE} theo. (from k_α)” estimated from the square root of (11), (12), (13), and with “ σ_{TIE} theo. (from σ_e)” estimated from the square root of the expression using the variance of the residuals such as (20).

Table 3 shows the same type of results but limited to the cesium clocks and with a linear interpolation.

Here also, the agreement between measurements and estimates is quite good.

On the other hand, Table 2 and 3 confirm that the standard deviation of the residuals is higher for a linear interpolation than for a quadratic one, whereas it is the opposite for the standard deviation of the TIE.

CONFIDENCE INTERVALS FOR THE INTERPOLATION AND EXTRAPOLATION ERRORS

Figure 2 shows that the real TIE may be outside the bounds of the estimated $\hat{\sigma}_{TIE}$. Thus, it is important to know the probability for the real TIE to be inside or outside these bounds. Moreover, it may be useful to improve the TIE bounds by using confidence intervals. This may be achieved by studying the statistics of the TIE.

We want to obtain a coefficient c_β ensuring the following confidence interval:

$$-c_\beta \cdot \hat{\sigma}_{TIE} < \text{TIE}(T_m, T_p) < +c_\beta \cdot \hat{\sigma}_{TIE} \quad \text{with} \quad \beta\% \text{ of confidence.} \quad (21)$$

By definition, c_β follows a Student distribution with ν degrees of freedom [10]. Consequently, for building the confidence interval (21), we have to use the Student coefficients for c_β . These coefficients are given in tables [10].

However, it is then necessary to know the number of degrees of freedom of the Student distribution, or, and this is equivalent, the number of degrees of freedom of the χ^2 distribution followed by $\langle \text{TIE}^2(T_m, T_p) \rangle$. Obviously, this number depends on how $\langle \text{TIE}^2(T_m, T_p) \rangle$ is estimated, i. e. from the variance of the residuals or from the noise levels k_α .

Estimation From the Variance of the Residuals

Equation (20) shows that $\langle \text{TIE}^2(T_m, T_p) \rangle$ is the product of a random variable $\hat{\sigma}_e^2$ by a constant number. Thus, the distribution of $\langle \text{TIE}^2(T_m, T_p) \rangle$ is the same as the distribution of $\hat{\sigma}_e^2$, i. e. a χ^2 distribution.

From Monte-Carlo simulations, we observed that the degrees of freedom of the distribution of $\hat{\sigma}_e^2$ only depend on the type of noise but, curiously, neither on the number of data N , nor the sampling period τ_0 .

We measured the following degrees of freedom ν for the χ^2 distribution of $\hat{\sigma}_e^2$:

- White FM: $\nu \approx 8$;
- Flicker FM: $\nu \approx 3$;
- Random walk FM: $\nu \approx 2$.

| $S_x(f)$ | ν | $c_{70\%}$ | $c_{95\%}$ |
|----------------|-------|------------|------------|
| $k_{-2}f^{-2}$ | 8 | 1.1 | 2.3 |
| $k_{-3}f^{-3}$ | 3 | 1.2 | 3.2 |
| $k_{-4}f^{-4}$ | 2 | 1.4 | 4.3 |

Table 4: Degrees of freedom of $\hat{\sigma}_e^2$ and $\hat{\sigma}_{TIE}$, and confidence coefficients c_β for 70% and 95% versus the types of noise [10].

Therefore, the c_β Student coefficient must be chosen with these degrees of freedom, according to the type of noise. Table 4 gives these coefficients for a 70% confidence interval (1σ) and for a 95% confidence interval (2σ).

Large Estimation From the Noise Levels

In this case, the degrees of freedom of $\langle TIE^2(T_m, T_p) \rangle$ are equal to the ones of the estimated noise level \hat{k}_α and then depend on the accuracy of its estimation. For instance, if \hat{k}_α was estimated by using the Allan variance, its degrees of freedom depend on the length of the sequence and on the number of sample used by the Allan variance [11, 12]. If this sequence is long enough, the degrees of freedom may be much greater than the values obtained from the variance of the residuals. The degrees of freedom have to be estimated at each noise level measurement [11, 12].

Therefore, if the noise levels are precisely determined, the estimation of $\hat{\sigma}_{TIE}$ is far better by using this method than from the variance of the residuals.

CONCLUSION

The first application of this work may be the selection of clocks according to their time stability performances. We may fix a limit for the maximum acceptable σ_e and TIE for a given interpolation sequence and extrapolation time. Let us consider that for an interpolated period $T_m=24h$ and for an extrapolated time $T_p=3.5h$, we fix : $\sigma_e < 2.1$ ns and $TIE < 5ns$. The use of Equations (8) to (13) allows us to translate the above specifications into specifications on the noise levels k_α (for instance, for a random walk FM, these specifications become $k_{-4} < 3.7 \cdot 10^{-33}s^{-1}$). Furthermore, these specifications may be translated again in term of Allan variance over 1 day (in the case of the random walk FM, it yields $\sigma_y(24h) < 3 \cdot 10^{-13}$). Thus, we can obtain a very simple criterion by using the Allan variance, ensuring that the specifications for σ_e and TIE will be respected.

Besides the interest of this method for navigation satellite systems, it may be used for defining a new method for very-long-term stability analysis.

A clock may be continuously measured during a few days (e.g. a time deviation measurement with a sampling period of 1 minute during 10 days). From these data, the noise levels of this clock could be precisely determined and a quadratic fit could be carried out. Thus, if the clock is continuously running in the same conditions, it could be possible to extrapolate the difference of this clock with the parabolic fit after a few months or 1 year.

This analysis could be helpful to low-accuracy purposes of time keeping, for instance for industrialists who periodically send their clock to an accreditation laboratory, or for applications which need a large autonomy.

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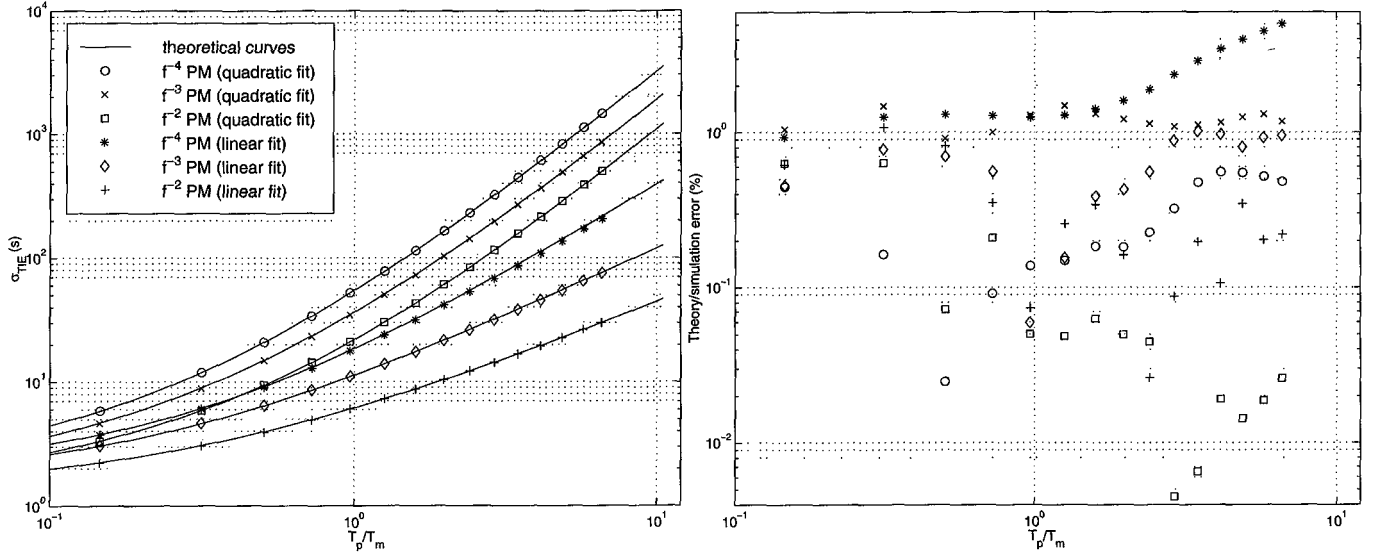


Figure 1: Comparison of the estimation of the standard deviation of the TIE calculated from the equation (11), (12), (13), (17), (18), (19) (solid lines) and estimated over 10000 realizations of simulated noise for a quadratic interpolation (\circ , \times , and \square) and for a linear interpolation ($*$, \diamond and $+$). In order to use the same scale, the noise levels were defined in such a way that the variance of the residuals is equal to one ($k_{-2} = 1.4 \cdot 10^{-4}s$, $k_{-3} = 3.3 \cdot 10^{-8}$ and $k_{-4} = 5.0 \cdot 10^{-12}s^{-1}$ for the quadratic interpolation, $k_{-2} = 3.5 \cdot 10^{-3}s$, $k_{-3} = 4.8 \cdot 10^{-7}$ and $k_{-4} = 3.3 \cdot 10^{-11}s^{-1}$ for the linear interpolation). The error bars corresponding to the estimates of the simulated noises are too small to be plotted on this graph. The right plot shows the differences between the theoretical curves and the simulation points: the larger error is equal to 5% but most of them are below 1%.

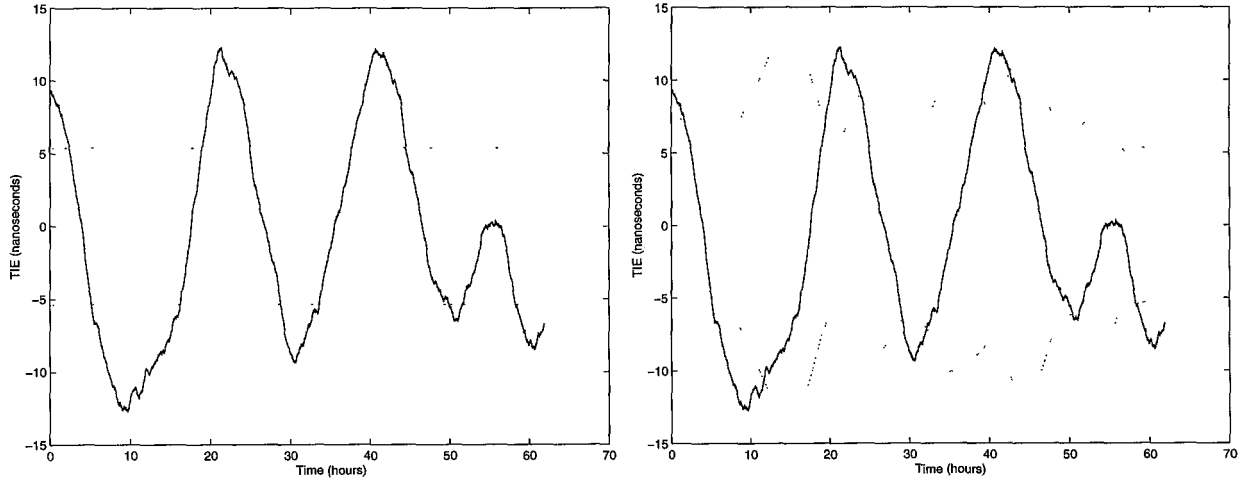


Figure 2: Estimation of the TIE for an ultra-stable quartz oscillator. The fit was performed over a 1-day sliding window. The TIE was measured 3.5 hours after the fitted sequence (solid line). The TIE bounds (dashed lines) were estimated from the noise levels (left) or from the variance of the residuals (right).